We review some general statistical properties of wave transport through surface disordered waveguides. These systems are shown to present both striking similarities and differences with respect to quasi-one-dimensional waveguides with volume disorder. The statistical properties are analysed using extensive numerical calculations and random matrix theory results. The transport properties are characterized by the statistical behaviour of different transport coefficients that can be defined for both classical (light, microwaves, sound, etc.) and quantum (electrons) waves. In analogy with bulk-disordered systems, the behaviour of the waveguide conductance/resistance (defined for both classical and quantum waves) as a function of the system length defines three different transport regimes: ballistic, diffusive and localization. However, the coupling between waveguide modes presents significant differences with respect to the coupling induced by volume defects. For any incoming mode, there is a strong preference for the forward propagation through the lowest mode. For narrow waveguides, the statistics of reflection coefficients (reflected speckle pattern) present strong finite-size effects which can be surprisingly well described by random matrix theory. Special attention is paid to the fundamental problem of the transition between different regimes. The long-standing problems of the phase randomization process between ballistic and diffusive regimes and the evolution of the conductance statistical distribution in the transition from diffusion (Gaussian statistics) to localization (log normal statistics) are also discussed.

1. Introduction

The statistical properties of coherent wave transport through disordered media has long been a subject of interest [1–11]. In the intermediate length scale or mesoscopic regime [12, 13] the inelastic scattering length can be much longer than the size of the system, and the waves can therefore maintain phase coherence as they travel the length of the system. Elastic scattering does not destroy phase coherence, which is why the effects of wave interference can modify the transport properties of disordered media. This is typically the regime of diffusive transport, characteristic of turbid media (for light waves) or disordered metals (electron waves).

In the last twenty years much attention has been focused on this subject in connection with the discovery of interesting multiple scattering effects in wave transport through random media [1–11]. Different phenomena such as the enhanced coherent backscattering [14–18] and intensity correlations in transmitted and reflected electromagnetic waves [19–31] are closely
connected to weak localization [32–34] and universal conductance fluctuations (UCF) [35–40] in electron transport. Recently there has been a renewed interest in this subject in connection with disorder effects in photonic crystals [41, 42].

These interesting phenomena are not restricted to the case of volume scattering. Enhanced backscattering [43–48] and correlations of the speckle pattern intensity [49–52] arising in the light scattering from randomly rough surfaces have been intensively investigated.

In addition to interference phenomena, it is also possible to observe size effects in the response functions because of the wave confinement in systems of small spatial dimensions. The effect of the surface on the transport properties of thin films has long been studied in the classical size effect regime [53, 54]. Classical size effects arise when the thickness of the film approaches the bulk mean free path, and have largely been interpreted according to the theories of Fuchs [55] and Sondheimer [56]. Most of the extensions of the classical Boltzmann-like approaches to wave-like transport through waveguides or quasi-one-dimensional (Q1D) disordered systems have been focused on the consequences of inhomogeneities and defects randomly distributed over the whole sample (bulk disorder) [57–65].

1.1 Scattering from rough surfaces

In many interesting practical cases, the waves propagate in a homogeneous medium where the main source of scattering is the interaction with rough or disordered surfaces. For classical waves this situation arises naturally in the study of optical waveguide fibres [66–68], sound waveguide lines, remote sensing, radio wave propagation, sonar, geophysical probing, etc. [2, 69, 70]. In electronic systems, it can be readily achieved by lateral confinement of a two dimensional electron gas 2DEG [6, 12, 13, 71, 72], surface states localized in stepped surfaces [73] or in metallic or molecular nanowires [74–77]. Due to the high quality of crystal growth methods and fabrication techniques it is commonly believed that at low temperatures, electron scattering is mainly caused by surface and interface disorder [78–80].

From a classical (particle scattering) point of view, the effects of scattering by surface inhomogeneities should be qualitatively different from the scattering by bulk impurities. In the absence of bulk inhomogeneities, the classical Boltzmann-like approaches predict a vanishing resistivity [6]. The waves with momentum nearly parallel to the channel walls can propagate over large distances without collisions and therefore short out the current flow. However, wave interference effects were shown to induce a non-zero surface resistivity in thin films [81–83] which, for large sample lengths, could even lead to strong localization effects [81, 82]. In analogy with bulk disordered systems, it is then possible to define different transport lengths such as mean free path or the localization length [81–84]. From the wave scattering point of view, naïve considerations would then suggest that the results for bulk disorder should be valid for the surface case after proper re-scaling. In this work we will see that the situation is different and much more complicated.

The peculiar properties of transport through surface disordered waveguides have motivated an increasing theoretical interest in the study of different statistical aspects of the problem during the last few years [84–98]. The analysis of these complex statistical problems usually requires the use of powerful numerical methods. Numerical approaches [86, 90, 98] based on the invariant embedding equation formulation [99–101] as well as rough surface ‘tight-binding’ calculations [59, 88] have been applied to the study of surface disordered waveguides. In this work we will present the results of extensive numerical calculations based on a mode-matching technique together with a generalized scattering matrix method [74, 92, 102].
The standard theoretical description of the above-mentioned problems is usually based on different perturbative treatments, [81–83, 89, 99–101, 103–105]. In contrast with these microscopic approaches, random matrix theory (RMT) provides a macroscopic approach since it deals with the statistical distribution of the transfer matrix for the full system [11, 106–109]. Most of the work based on RMT has been focused on the study of multiple-scattering effects in wave transport through bulk disordered media [11, 22, 107–112]. As we will see, some of the statistical properties of transport through surface-disordered waveguides are surprisingly well described by a RMT approach.

1.2 Outlook

The purpose of the present work is to discuss some recent advances in the understanding of general statistical properties of wave transport through quasi-one-dimensional systems (e.g. waveguides, nanowires, etc.) with surface roughness. We will discuss the main differences and similarities between the statistics of different transport coefficients for surface disorder and the analogous quantities in bulk disordered systems. We have not attempted to give a comprehensive list of references to theoretical and experimental work on this multidisciplinary subject and we apologize for those contributions we have overlooked.

In section 2 we introduce some general properties of the scattering matrix $S$ as well as the definition of different transport coefficients. The numerical results presented in this work are based on a simple model system which is presented in section 3. The analogies between electron transport and propagation of diffuse light through surface corrugated waveguides are discussed in section 4, where we introduce the optical analogue of the resistance. In full analogy with bulk disordered systems, the behaviour of the averaged resistance (conductance) as a function of the length of the system defines three different transport regimes: ballistic, diffusive and localized. The transport mean free path $\ell$ and the localization length $\xi$ are shown to be related by $\xi \approx N \ell$ (N being the number of channels) in agreement with the expected behaviour for bulk disordered systems. The consequence of time reversal symmetry breaking on the properties of $\ell$ and $\xi$ is also analysed by applying a magnetic field on a surface-disordered electron waveguide. The behaviour of the first moments of the different transport coefficients as a function of the length of the system is discussed in sections 4 and 5. The assumption of diffuse incoming waves in section 4 limits the study to those systems in which the incident modes are mutually incoherent. In section 5 we address the more practical situation in light propagation where a single incident mode couples with different forward and backward modes. As we shall see, the scattering from rough, perfectly conducting walls induces a strong and rich coupling to forward and backward modes. The coupling characteristics present significant differences with respect to the coupling induced by volume defects. For any incoming mode, there is a strong preference for the forward propagation through the lowest mode. The coupling to backward modes presents very interesting behaviour which is dependent on the incident mode. The ‘external’ modes (defined as those propagating modes with either the smallest or the largest transversal momentum) present an enhanced backscattering factor larger than for the other modes. In the case of the lowest mode, this factor exhibits remarkable oscillations as a function of the wavelength.

As the length of the system increases, the fluctuations of the transport coefficients become more and more important. As the length becomes of the order of or larger than the localization length, the fluctuation/mean-value ratio diverges exponentially. The statistical analysis then requires the knowledge of the full distribution functions. The statistical distributions of transmission and reflection coefficients are discussed in sections 6 and 7 respectively. Although the transport is strongly non-isotropic, the analysis of the probability distributions confirms the
surface-disordered configuration distributions predicted by random matrix theory for volume
disorder. Finally we will address different aspects of the transition between different regimes. In
section 8 we deal with the problem of phase randomization occurring in the transition from
ballistic to diffusive regimes. We will see that the statistical distributions of phases and in-
tensities are qualitatively similar to those observed in microwave tubes with bulk disorder
[113]. In section 9 we will analyse in detail the conductance distributions in the transition
from diffusive to localized regimes.

2. General aspects of wave propagation in quasi-one-dimensional wires

In this work we restrict ourselves to the problem of scalar waves following the standard wave
equation

$$\nabla^2 \Psi + k^2 (1 + \delta n(r)) \Psi = 0, \quad (1)$$

where, for classical waves, $k$ is the wavenumber ($k \equiv \omega/c$) and $1 + \delta n(r)$ is the refraction
index. For quantum waves, $k^2 \equiv (2m/\hbar^2)E$ and $n(r) \equiv 1 - U(r)/E$, where $E$ is the energy of the
particle and $U(r)$ the potential energy. In this section we will discuss some general properties
of the scattering of scalar waves in quasi-one-dimensional systems, i.e. the waves are confined
in the lateral direction ($x$) whereas transport occurs along the $z$ direction. A standard sketch
of the problem is shown in figure 1. We consider the transport problem through a scattering
region of length $L$ connected to two perfect leads or waveguides. Inside the perfect leads
(where $n_r = n_r(x)$ does not depend on $z$) the lateral confinement defines a set of wave modes

$$\Psi_n^\pm = \frac{1}{\sqrt{k_n}} \psi_n(x) e^{\pm ik_n z}, \quad (2)$$

where $\{\phi_n\}$ is a complete set of orthonormal transversal eigenfunctions ($\nabla^2 \phi + (q_n^2 + k^2 \delta n(x))\phi = 0$) and $k^2 = q_n^2 + k_n^2$. The integer $n = 1, 2, \ldots$ labels the wave modes, also
referred to as scattering channels. Those modes having a real longitudinal wavenumber $k_n = \sqrt{k^2 - q_n^2}$ are called propagating modes and those modes having imaginary lines are
called evanescent modes. The normalization of the total wave function $\Psi_n$ is chosen in such
a way that propagating modes carry unit current (power):

$$\int dx \Im \{\Psi_n^+ \nabla \Psi_n^-\} = \pm \delta_{nm}. \quad (3)$$

A wave incident on the scattering region can be described on this basis by a vector of
coefficients

$$(i^L, i^R) = (i^L_1, i^L_2, \ldots, i^L_{N_L}, i^R_1, i^R_2, \ldots, i^R_{N_R}). \quad (4)$$

Figure 1. Sketch of the system defining the $S$ matrix.
The first $N_L$ coefficients describe the signal arriving from the left lead and the following $N_R$ coefficients correspond to the wave arriving through the right lead in figure 1.

Similarly, the outgoing wave (i.e. the reflected and transmitted waves) can be described by the vector of coefficients
\[
(o^L, o^R) = (o_1^L, o_2^L, \ldots, o_{N_L}^L, o_1^R, o_2^R, \ldots, o_{N_R}^R).
\]

### 2.1 Scattering matrix and transport coefficients

The scattering matrix ($S$ matrix) was introduced by Heisenberg to describe a scattering process without any assumption about the details of the interaction [114, 115]. In this formulation, the process is thought of as a transformation of an incoming state into an outgoing state, which describe the system far away from the interaction potential. Hence, the $S$ matrix relates the asymptotic propagating outgoing waves $(o^L, o^R)$ to the incoming ones $(i^L, i^R)$
\[
\begin{pmatrix}
  o^L \\
  o^R
\end{pmatrix} =
\begin{pmatrix}
  r \\
  t
\end{pmatrix}
\begin{pmatrix}
  i^L \\
  i^R
\end{pmatrix},
\]

where the reflection block $r$ ($\tilde{r}$) is a square $N_L \times N_L$ ($N_R \times N_R$) matrix, while the transmission block $t$ ($\tilde{t}$) is a $N_R \times N_L$ ($N_L \times N_R$) matrix.

The matrix elements $r_{ba}$ and $t_{ja}$ denote the reflected amplitudes in channel $b$ and the transmitted amplitudes resulting in channel $j$ when there is a unit flux incident from the left in channel $a$; $\tilde{r}_{ji}$ and $\tilde{t}_{bi}$ have an analogous meaning, except that the incident flux in channel $i$ comes from the right.

For a given incoming mode $a$ the transmission ($T_{aj}$) and reflection ($R_{ab}$) coefficients are defined as
\[
R_{ab} \equiv |r_{ba}|^2 \\
T_{aj} \equiv |t_{ja}|^2.
\]

The *speckle pattern* observed in disordered samples is just the complex interference pattern $T_{aj}$ as a function of the outgoing direction $j$.

The total transmission and reflection for the incoming mode $a$ are given by:
\[
T_a = \sum_{j=1}^{N_L} T_{aj} \\
R_a = \sum_{b=1}^{N_R} R_{ab}.
\]

For simplicity, in the rest of this work we will consider $N_L = N_R = N$.

It is well known that the $S$ matrix exhibits some properties that are independent of the specific problem under study. In particular, it is unitary ($S^\dagger S = 1$) and symmetric ($S = S^\top$), with these two properties reflecting probability (or energy) conservation in elastic scattering and reciprocity, respectively [116, 117]. (For a discussion of the relation between time-reversal invariance and reciprocity, see [117].) The general aim is to get maximum information about the $S$ matrix with minimum knowledge about the interaction itself. Other properties may be derived based, for example, on dispersion relations and causality conditions [118]. The existence of such general properties of the $S$ matrix is the reason why it has become a fundamental tool in most areas of theoretical physics. This formulation has also found a wide range of applications with the development of random matrix theory [119], which has recently acquired renewed
interest through its use in quantum- and classical-wave transport in random media [11, 109]. For a simple application of RMT results see Appendix A.

The S matrix was originally defined as an operator acting on asymptotic states. However, it is possible to define a generalized or extended S matrix including evanescent fields [11, 117, 120] which is not unitary but exhibits a set of extended unitary conditions (known as generalized Stokes relations in optics [117]). This generalized scattering matrix (GSM) has played an important role in the development of powerful numerical methods to calculate wave transport in waveguide geometries [74, 102] and it plays a key role in the numerical results discussed in this work.

2.2 Conductance and transport eigenchannels

The seemingly unrelated problems of classical wave transport in waveguides and electronic transport in mesoscopic conductors are tied together by the Landauer formula [40]. The dimensionless conductance $G$ of a waveguide can be defined, both for classical and quantum waves, as the sum over transmission coefficients $T_{aj}$ connecting all input modes $a$ and output modes $j$, $G = \sum a_j T_{aj}$. Since $S$ is unitary, we also have

$$G = \sum_{aj} T_{aj} = \text{Trace}\{t^\dagger t\} = \sum_{ja} \tilde{T}_{ja} = \text{Trace}\{\tilde{t}^\dagger \tilde{t}\},$$

(9)

$$G = N_L - \sum_{ab} R_{ab} = N_L - \text{Trace}\{r^\dagger r\} = N_R - \sum_{ij} \tilde{R}_{ij} = N_R - \text{Trace}\{\tilde{r}^\dagger \tilde{r}\}.$$  

(10)

In electronic systems at low bias, the conductance is determined by the electrons at the Fermi level flowing from one electrode to the other. In Landauer’s picture, the electrodes are electron reservoirs and all the incoming modes are statistically equivalent, electrons flowing in different channels are mutually incoherent. The electronic conductance is given by the Landauer’s formula

$$G_e = \frac{2e^2}{h} \sum_a T_a,$$

(11)

where $2e^2 / h$ is the quantum of conductance ($e$ being the electron charge and $h$ the Planck’s constant). For a perfect (transparent) waveguide, $N_L = N_R = N, T_a = 1$ and the conductance is simply given by the number of propagating modes. The conductance will then change stepwise by changing the wavelength or the waveguide cross-section (i.e. by changing the (integer) number of modes). This is the well known phenomenon of conductance quantization in electronic systems: the conductance of a perfect wire is an integer multiple of the quantum of conductance ($2e^2 / h$) [71, 72].

To resemble this picture for classical waves (e.g. light or microwaves) we could excite the waveguide modes with random monochromatic waves. This can be achieved, for example by previously passing an incident monochromatic wave through a moving diffuser so that correlation between phases of different modes is lost. In this way, each mode carries the same time averaged power and the normalized total transmitted power is given by the dimensionless conductance [121].

2.2.1 Eigenchannels. Let us consider that the asymptotic right and left leads are equivalent, i.e. $N_L = N_R = N$. As a consequence of unitarity, the four Hermitian matrices $t^\dagger t, \tilde{t}^\dagger \tilde{t}, r^\dagger r$ and $\tilde{r}^\dagger \tilde{r}$ have the same set of eigenvalues $T_1, T_2, \ldots, T_N$. Each of these $N$ transmission eigenvalues is a real number between 0 and 1. Transport eigenchannels, constructed by an appropriate linear combination of waveguide modes, can be seen as ‘extended states’ governing the transport
properties of mesoscopic systems. It can be shown [11, 107, 109, 122] that any $S$ matrix satisfying the flux conservation requirements can be written in terms of the $T_n$'s by means of the polar decomposition

$$S = \begin{pmatrix} u^{(1)} & 0 \\ 0 & u^{(2)} \end{pmatrix} \begin{pmatrix} -\sqrt{1-\tau} & \sqrt{\tau} \\ \sqrt{\tau} & \sqrt{1-\tau} \end{pmatrix} \begin{pmatrix} u^{(3)} & 0 \\ 0 & u^{(4)} \end{pmatrix},$$

where $u^{(i)} (i = 1, 2, 3, 4)$ are unitary matrices and $T = \text{diag}(T_1, T_2, \ldots, T_N)$ is a $N \times N$ diagonal matrix with the transmission eigenvalues on the diagonal. From reciprocity it immediately follows that the matrices $u$ satisfy that $u^{(1)} = (u^{(3)})^T$ and $u^{(2)} = (u^{(4)})^T$.

The transmission and reflection matrices can then be written as

$$r = -u^{(1)}(\sqrt{1-T})u^{(1)T},$$

$$t = u^{(2)}(\sqrt{T})u^{(1)T},$$

or

$$r_{ba} = -\sum_n u^{(1)}_b(\sqrt{1-T_n})u^{(1)T}_n,$$

$$t_{ja} = \sum_n u^{(2)}_j(\sqrt{T_n})u^{(1)T}_n.$$

In general, the transmission matrices $t$ and $\tilde{t}$ are rectangular matrices (for $N_L \neq N_R$). The matrix product $t^\dagger t$ is then a square ($N_R \times N_R$) matrix, while $\tilde{t}^\dagger \tilde{t}$ is a square ($N_L \times N_L$) matrix. The two matrix products $t^\dagger t$ and $\tilde{t}^\dagger \tilde{t}$ contain a common set of $\min(N_L, N_R)$ nonzero transmission eigenvalues.

In analogy with molecular orbitals in atomic physics, transport eigenchannels are the natural basis to analyse electronic transport in nanostructures [123, 124]. Experimental information about the transport eigenvalues is not accessible by usual conductance measurements. However, by using the nonlinear current-voltage characteristics of superconducting constrictions it is possible to measure the transport eigenvalues of nanometer scale contacts [125, 126]. The concept of transport eigenchannels will play a key role in our analysis of the conductance distributions in the transition from diffusive to localization regimes.

3. Model system and numerical methods

In order to analyse the statistical properties of transport through surface disordered waveguides, we have performed extensive numerical calculations based on the simple two-dimensional (2D) model system sketched in figure 2. The waveguide is assumed to have perfectly reflecting walls.

![Figure 2. Schematic view of the model system under consideration.](image-url)
The scattering region consists of a central region of length $L$ with one of its walls corrugated. For electromagnetic waves, we will assume s-polarization with the electric vector parallel to the grooves (TE modes).

The scattering region is attached to two perfect leads of width $W_0$ where the confinement defines a set of transverse eigenfunctions

$$\psi_n(x) = \sqrt{\frac{2}{W_0}} \sin(q_n x), \quad q_n = \frac{n\pi}{W_0},$$

and $N$ propagating waveguide modes

$$\psi_n^\pm = \frac{1}{\sqrt{k_n}} \psi_n(x) e^{\pm ik_n z},$$

with

$$k^2 \equiv \frac{2\pi}{\lambda} = k_n^2 + q_n^2 = \frac{n^2 \pi^2}{W_0^2} + \left(\frac{n\pi}{W_0}\right)^2, \quad N = \text{Integer}\left\{\frac{2W_0}{\lambda}\right\}. \quad (17)$$

The corrugated part of the waveguide, of total length $L$, is composed of $n$ slices of length $l$. The width of each slice has random values uniformly distributed between $W_0 - \delta$ and $W_0 + \delta$ about the mean value $W_0$. We will restrict ourselves to relatively weak roughness values ($l, \delta \ll \lambda$). We shall take $W_0/\delta = 7$ and $l/\delta = 3/2$ (i.e. $2\delta/W_0 = 0.286$ and $l/W_0 = 0.214$).

The main transport properties do not depend on the particular choice of these parameters as long as the wavelength of the incident radiation remains larger than the length $l$ of each individual slice [127].

Transmission and reflection coefficients are exactly calculated on solving the 2D wave equation, by mode matching at each slice, together with a generalized scattering matrix technique [74, 92, 102, 128].

In order to analyse the influence of surface disorder on the transport properties we will perform an extensive statistical study. Ensemble averages, denoted by $\langle \rangle$, are performed as follows:

(i) For a fixed value of parameters the width of each slice is randomly chosen as already mentioned above.
(ii) For that choice of parameters the full scattering matrix is obtained.
(iii) From the elements of the scattering matrix, transmission and reflection coefficients are computed.
(iv) The information for this first realization of the disorder is stored.
(v) Go back to the first step.

4. Transport regimes in surface disordered waveguides

In analogy with bulk disordered systems we can define three different transport regimes, all of them given by the relationship between the length of the disordered part of the system and the transport mean free path $\ell$ or the localization length $\xi$ (see figure 3). (There is also another important length scale in the system, the phase coherence length $l_\phi$, which equals the average distance that the wave diffuses without losing phase coherence. Here we will assume it is much longer than any other length.) We should remark that when bulk disorder is present, the width of the waveguide also plays a role, and intermediate regimes can be defined since lateral diffusion may take place.
Figure 3. The three main transport regimes: ballistic $L < \ell, \xi$; diffusive $\ell < L < \xi$; and localization $\ell, \xi < L$.

- **Ballistic regime, $L < \ell, \xi$**
  In the ballistic regime the length of the disordered part is such that neither impurity nor boundary scattering significantly disturb the wave propagation through the system.

- **Diffusive regime, $\ell < L < \xi$**
  This regime corresponds to intermediate lengths, smaller than $\xi$ but larger than $\ell$. In this regime Ohm’s law holds and thus the resistance grows linearly as a function of $L$. It should be noted that we are always talking about elastic scattering and thus there is no energy loss in the scattering processes. For a clean ($L = 0$) waveguide the normalized conductance is given by the number of propagating modes $G = N$, therefore the resistance is $R = 1/N$. As the length increases, averaging over several configurations of the disorder results in an increase of the resistance corresponding to Ohm’s law, therefore:

$$R(L) = \frac{1}{G} \approx \frac{1}{N} + \frac{L}{N\ell}.$$ (18)

This has been numerically tested for various systems using a tight-binding based formulation [57–61, 63, 64] in the case of volume disorder. Notice that the mean free path in equation 18 differs in a numerical prefactor from the transport mean free path $\ell^*$ of the kinetic theory ($\ell/\ell^* = 2$ in 1D systems, $\ell/\ell^* = \pi/2$ in 2D systems and $\ell/\ell^* = 4/3$ in 3D systems) [109].

- **Localization regime, $\ell, \xi < L$**
  The localization regime is a consequence of quantum interference effects. In this regime the wave interference is such that the wave propagates along the system and, eventually, once it travels a distance of the order of $\xi$, becomes completely trapped. For larger lengths than $\xi$ the transmission along the system decreases exponentially and thus, deep inside the localization regime, the resistance increases exponentially [57, 58, 60]. The fluctuations associated either with the resistance or with the conductance are so large that the average of neither the resistance nor the conductance is a good measurement of the transport properties. It can be shown that the appropriate quantity to address transport properties in the localization regime is the logarithm of either the conductance, $\ln(G/G_0)$, or the resistance, $\ln(RG_0)$. Therefore, after averaging over many configurations, $\langle \ln(G/G_0) \rangle$ decreases linearly as a function of the length of the system, i.e. $\langle \ln(G/G_0) \rangle \propto -L/\xi$. 

Figure 4. (a) $\langle R \rangle$ versus the length $L$ of the fit corrugated region. The straight line represents the best fit of the linear behaviour associated with the diffusive transport regime (see equation 19). The results are obtained for $W_0/\lambda = 2.6$, i.e. they correspond to 5 propagating modes in the waveguide. The corresponding effective mean free path is $\ell/W_0 \approx 6.3$. (b) $\langle \ln(G) \rangle$ versus $L$ for the same case as in (a). The best fit of the linear part of the curve is also shown. The corresponding localization length is $\xi/W_0 \approx 34.4$. Reproduced with permission from [84], © American Institute of Physics.

4.1 Mean free path and localization length

The characterization of the different transport regimes can be carried out by analysing the dependence of $\langle R \rangle$ and $\langle \ln(G) \rangle$ on $L$ [84]. While in a clean system the transport is ballistic and consists essentially of unscattered waves (the mean free path $\ell$ is then much larger than both $L$ and $W_0$), in a randomly corrugated waveguide the two regimes, diffusive and localized, are clearly visible as shown in figure 4. This shows $\langle R \rangle$ and $\langle \ln(G) \rangle$ versus $L$ (figures 4(a) and 4(b), respectively), for $W_0/\lambda = 2.6$. As $L$ increases from zero, $\langle R \rangle$ follows first an almost perfect linear behaviour with $L$ which corresponds to the semi-ballistic [61] regime. In this case there is so much scattering in the longitudinal direction that the transport is diffusive, while in the transversal direction almost no scattering occurs ($W_0 < \ell < L$). Then, in analogy with Ohm’s law for electron wires, the waveguide ‘resistance’ can be described by an ‘ohm-like’ term proportional to $L/W_0$ plus a ‘contact resistance’ $R_c$ [60, 62, 64, 84],

$$\langle R \rangle = R_c + \rho \frac{L}{W_0} = R_c + \frac{L}{N\ell}, \quad (19)$$

where the ‘resistivity’ $\rho$ and the mean free path $\ell$ are related by $\ell/W_0 = 1/(N\rho)$ [60].

This linear behaviour breaks down as $L$ increases further. Then, a typical linear decrease of $\langle \ln(G) \rangle$ with $L$ appears, as shown in figure 4(b). The system has now entered the localized regime at which $W_0 < \ell \ll L$. The localization length $\xi$ is defined by [39, 129, 130]

$$\xi \equiv -\left(\frac{\partial \langle \ln(G) \rangle}{\partial L}\right)^{-1}. \quad (20)$$

The contact resistance $R_c$ and the effective mean free path $\ell/W_0$ can easily be obtained both from equation 19 and by a least-squares fit of the linear part in figure 4(a).

Optical waveguides allow us to analyse the behaviour these quantities exhibit when the energy (thus the wavelength) of the incoming wave changes. In a clean system a plot of
Wave transport through waveguides

Figure 5. Averaged transmittance $\langle G \rangle$ as a function of $W_0/\lambda$ for fixed values of $\delta/W_0$ and $l/W_0$, $(W_0 = 7\delta$ and
$l = (3/2)\delta$). Note that the case $L/W_0 = 0$ corresponds to a flat waveguide. Reproduced with permission from [84], © American Institute of Physics.

$\langle G \rangle$ versus $W_0/\lambda$ (see figure 5) shows a staircase behaviour which constitutes the analogue for classical waves of the conductance quantization of electronic systems [121]. Figure 5 also contains $\langle G \rangle$ versus $W_0/\lambda$ for different length values $L$ of the corrugated portion of the waveguide. For moderate-length waveguides, the transmittance shows a dip just at the onset of a new propagating mode. As the length $L$ of the corrugated region increases, $\langle G \rangle$ presents (see figure 5) an oscillatory behaviour that, as we show below, reflects successive transitions from diffusive to localized regimes.

These successive transitions have their origin in the behaviour of both mean free path and localization length. Figure 6(a) shows how the mean free path oscillates as $W_0/\lambda$ increases, having its minima close to half-integers of $W_0/\lambda$, which correspond to the appearance of a

Figure 6. (a) Effective mean free path $\ell/W_0$ versus $W_0/\lambda$. The inset shows the behaviour of the contact resistance $R_c$ together with the expected value for a perfect waveguide (dotted line). (b) Localization length $\xi/W_0$ (filled circles) and $N\ell$ (open circles) versus $W_0/\lambda$. $\xi \approx N\ell$, within the numerical accuracy. Reproduced with permission from [84], © American Institute of Physics.
new propagating mode in the waveguide. The correlation between the minima in figures 5 and 6 is evident and both are the result of an increase of the scattering events due to the mode coupling with the not yet propagating new mode.

Similar dips appear in the case of scattering by volume defects [61, 63, 131–134], whose origin is pretty much the same: the existence of a quasi-bound state due to the presence of an impurity [135, 136].

The dependence of the localization length $\xi$ on the wavelength (figure 6(b)) can also be obtained from the least-squares fit of the linear part of the plot of $(\ln(G))$ versus $L$ in figure 4(b). It should be noted that, within the numerical accuracy of our calculation, $\xi \approx N\ell$ as shown in figure 6(b). This again has an analogy in agreement with the expected behaviour for electron transport through disordered media [137, 138].

The wavelength dependence of both localization length and mean free path shows that, for a fixed length, it is possible to go from the localized regime into the diffusive one by changing the wavelength of the incoming wave.

4.2 Single surface defect

In order to discuss the origin of the oscillations exhibited by transmittance, mean free path and localization length, let us analyse the effect that a single defect has on the transmittance.

In figure 7 we have plotted the total transmittance for a waveguide with (a) one single defect of height $\delta = W_0/14$ at one edge making the waveguide wider (upward defect), (b) one single defect (same $\delta$ as in (a)) making it thinner (downward defect) and (c) the result of averaging over 100 realizations of a waveguide with $L = 13l$ and with the width of each slice randomly distributed between $W_0 - \delta$ and $W_0 + \delta$.

Figure 7. (a) Total transmittance $G$ versus $W_0/\lambda$ for a waveguide of width $W_0$ with a defect of height $\delta = W_0/14$ (the central part then has width $W' = W_0 + \delta$) and length $l = 3\delta$ (see the detail of the dip on the right figure). (b) The same as (a) but the central portion now has a width $W' = W_0 - \delta$. (c) Average over 100 realizations for a disordered system composed of 13 slices of length $l$ and random width between $W_0 - \delta$ and $W_0 + \delta$. 
At a glance we can see that the effects of an upward and of a downward defect on the transmittance are completely different from each other. While the former produces a dramatic decrease of the transmittance for values of $W_0/\lambda$ at the onset of a new propagating mode, the latter leads to an overall smoothing of the curve $G$ versus $W_0/\lambda$. Interestingly, the behaviour of the curve corresponding to the average over several configurations seems to respond to a mixture of these effects: deep minima appear close to the values where the upward defect produces the biggest effect and also a smooth increase between the dips corresponds to the smoothing that has arisen from the effect of a downward defect.

The origin of the dips lies in the appearance of a new propagating mode in the ‘disordered part’ which is yet evanescent in the clean parts (i.e. it is a spatially localized state). When this is satisfied, the incoming modes can scatter into this mode and eventually the wave is completely reflected back. In figure 8 we have plotted the intensity along a one-channel waveguide ($W_0/\lambda = 0.92$) with an upward defect with two different lengths. In the left panel the length of the defect is such that the total transmittance is close to one ($l = 0.7W_0$), whereas in the other panel it is close to zero ($l = 1.3W_0$). On the left hand side it is clearly visible how the wave propagates from left to right with almost no perturbation. In the right panel the wave is entirely reflected back, having a large maximum at the defect. The strong coupling of the ‘propagating’ (first) mode in the left and the ‘localized’ (second) state in the defect can clearly be seen in the figure. Consequently the appearance of the dip is a compromise between the width and the length of the defect and the wavelength.

### 4.3 Breaking of time reversal invariance

All the phenomena above have been described assuming that reversed paths are equivalent, i.e. time reversal invariance holds. For electronic transport, this intrinsic symmetry can be lifted by applying a magnetic field to the system. Let us now discuss the behaviour of important quantities under the effect of a magnetic field that removes time reversal invariance.

The consequences of time reversal symmetry breaking in disordered media, such as the doubling of the localization length because of the phase mismatch of time reversed paths, have been conceptually known for a long time (e.g. [4, 5, 7, 8, 109]). Nevertheless, a fundamental question is still under debate: how does the transition take place? Previous theoretical [139] as well as numerical works, by means of an Anderson model (bulk disorder) [140], have proposed

![Figure 8](image_url)  
Figure 8. (Left) Total intensity for an upward defect ($W_0/\lambda = 0.92$, $l = 0.7W_0$ and $\delta = W_0/7$) leading to an almost unperturbed outgoing wave. (Right) The same as (a) but with $l = 1.3W_0$ giving rise to a totally reflected wave.
that the expected doubling of the localization length takes place in a rather smooth way, while the mean free path remains constant.

Here, we are going to discuss the consequences that applying a weak magnetic field perpendicular to a surface disordered wire has on the values of the localization length and on the mean free path. To introduce the magnetic field into the scattering matrix formulation we have followed the method given in [141].

Figure 9 shows the usual plot \(\langle \ln G \rangle\) versus \(L/W\), but now for different values of the magnetic field, \(B\). For convenience the magnetic field strength is given by the quantity \(\Phi_1 = \xi_0 WB\) [95, 140] that measures the amount of magnetic flux trapped in a portion of wire with a length equal to the localization length at zero field (\(\xi_0\)). By inspection of figure 9 it can be seen that the larger the magnetic field the smaller the slope of \(\langle \ln G \rangle\) versus \(L/W\), giving rise to an increase of the localization length.

However, figure 9 contains more information. In particular, the plot of \(\langle R \rangle\) versus \(L/W\) shows that the slope is also modified by the presence of the magnetic field. This indicates that, for surface disordered wires, the mean free path is also affected by the presence (or absence) of time reversal invariance [95].

The expected behaviour of the localization length as a function of the magnetic field turns out to be rather different to that expected (see figure 10). Instead of reaching the saturation value \(\xi_B = 2\xi_0\) [109, 139, 140, 142] the values go far beyond, i.e. \(\xi_B \gg 2\xi_0\). This brings an interesting question: does the RMT relation \(\xi = \beta N \ell\) hold? The answer is also in figure 10. Because the mean free path is also affected by the gradual lifting of time reversal invariance,

![Figure 9. Logarithm of the conductance (upper panel) and resistance (lower panel) as a function of the length of the disordered region for different values of the magnetic field (symbols). The lines indicate the best linear fits, which are used to extract the localization length (from the logarithm of the conductance), and the mean free path (from the resistance). Reproduced with permission from [95], © American Physical Society.](image-url)
Figure 10. Magnetic dependence of the mean free path and of the localization length. Both the mean free path and the localization length are normalized to their zero-field value; the magnetic field is in units of $W/L_B$, with $L_B = \sqrt{\hbar/\epsilon B}$, being the magnetic length. Inset: localization length divided by the number of modes and the mean free path; this quantity reaches the value 2 for fully broken time reversal symmetry. Reproduced with permission from [95], © American Physical Society.

it is natural to take its precise value for each magnetic field considered. The fact that both $\xi_B$ and $\ell_B$ increase smoothly when $B$ increases and do it in the same qualitative manner allows the possibility of having huge values of $\xi_B$ without violating $\xi = \beta N \ell$ in any case (see inset in figure 10).

The reason for that lies in the presence of one clean surface [95]. In fact, even for really small $B$, the developing edge states may propagate along the clean surface suffering almost no scattering.

The magnetic field dependence of the mean free path and the localization length permits a fine tuning of the onsets of the diffusive and localization regimes. Moreover, the extension of the diffusive regime can also be manipulated by acting over the value of the magnetic field.

5. Transmission and reflection coefficients

So far we have discussed some analogies between electron transport in metallic wires and propagation of diffuse light through surface corrugated waveguides. Nevertheless, we restricted the study to the mean values of the total transmittance of a diffuse incoming wave. Transport properties are not only characterized by these, but a detailed analysis of the behaviour of the coupling between incident and transmitted channels is required. This kind of analysis is not easily achieved in electronic systems, but in special cases [126], only a few numerical studies based on volume (Anderson-like) disorder have been performed [63, 65]. However, light or other classical wave propagation gives the perfect environment to address this question directly [143, 144].

As we will see, the scattering from rough, perfectly conducting walls induces a strong and rich coupling to forward and backward modes. The coupling characteristics present significant
differences with respect to the coupling induced by volume defects [63]. For any incoming mode, and when the length of the system is larger than the localization length $\xi$, there is a strong preference for the forward propagation through the lowest mode.

5.1 Averaged transmission coefficients

Here we perform the analysis of the transmission coefficients $T_{ij}$ as a function of the length of the system. Their behaviour reveals that there are qualitative differences between random surface and volume scattering [85, 86, 90]. To address this question we will take the same system as above, fixing the ratio between the waveguide width and the wavelength to $W_0/\lambda = 2.6$. With this choice, the waveguide allows five propagating modes.

Tight binding calculations [63] based on bulk disordered (Anderson) wires have shown that the transmission coefficients behave quite isotropically. In contrast, both $T_{ij}$ and $T_i$ in these surface disordered waveguides show a clear non-isotropic distribution, even in the diffusive regime ($\ell \leq L \leq \xi$). In figures 11(a) and (b) we have plotted the averaged transmission coefficients $\langle T_{ij} \rangle$ as a function of the normalized length of the disordered region. Calculations based on the solution of the invariant embedded equations [86, 90] for a different model of the surface roughness produce equivalent results to those presented in figure 11. The two vertical dashed lines are located at the mean free path and the localization length. For very small lengths perturbation theory gives $T_{ij} \approx \delta_{ij} - (2\delta_{ij} - 1)L/\ell_{ij}$ where $\ell_{ij}$ can be seen as an intermodal mean free path [90]. In view of the behaviour of the $\langle T_{ii} \rangle$ as a function of the length one could propose transport regimes that depend on the incoming mode (ballistic: $\langle T_{ii} \rangle \sim 1 - L/\ell_{ii}$, ...
diffusive: $\langle T_{ii} \rangle \sim \ell_{ii}/L$ and localization: $\langle \ln T_{ii} \rangle \sim -L/\xi_{ii}$). Therefore, for a fixed value of $L$ this would give rise to the coexistence of different transport regimes within the same sample [86, 90]. Notice that this ambiguity is not present when the transport regimes are defined from the behaviour of the resistance/conductance [84]. (In fact, a wavepacket inside a sample smaller that $\xi$ cannot become localized since at least one of the scattering channels do not decay exponentially.)

In the localization regime ($L > \xi$), there is a strong preference to couple with the lowest transmitted mode $j = 1$, irrespective of the incoming mode $i$. The coupling to the forward modes is well ordered, decreasing as the transversal momenta of the outgoing modes increase, and decreasing as the longitudinal momenta of the outgoing modes increase. For $T_i$ there is also a marked preference to forward propagation through the lowest mode as shown in figure 11(c). This is more clearly seen in figure 11(d) and (e) where we plot $\langle \ln(T_{ij}) \rangle$ as a function of the normalized length $L/W_0$ for two different incident modes $i = 1$ and $i = 3$ and $\langle \ln(T_i) \rangle$ versus $L/W_0$ for all the modes (figure 11(f)).

All curves present the same linear dependence at large $L$. The inverse of the slope defines a length $\xi_{ij} = -(\partial \langle \ln(T_{ij}) \rangle / \partial L)^{-1}$ which does not depend on the incoming or outgoing modes ($\xi_{ij}/W_0 = 34.4 \pm 0.7$ for the structure of figure 2). Exactly the same dependence is found for $\langle \ln T_i \rangle$. As a general result, and within numerical accuracy, $\xi_{ij} = \xi_i = \xi = N \ell$.

### 5.2 Averaged reflection coefficients

Let us now discuss the numerical results for the reflection coefficients in a surface disordered waveguide. In order to obtain the mean values as well as the intensity distributions for the reflection coefficients, we have performed the calculations over an ensemble of one thousand configurations.

Figures 12(a) and (b) show the averaged reflection coefficients $\langle R_{ij} \rangle$ as a function of the length $L$. The coupling to backward modes presents very interesting behaviour, showing a marked dependence on the incident mode. This can be associated with an anomalous enhanced backscattering (EB) factor. From studies developed in the context of RMT, it is predicted [22, 108, 109] that, in the presence of time reversal symmetry $\langle R_{ii} \rangle$ is twice as large as $\langle R_{ij} \rangle$. However, numerical calculations in waveguides with volume disorder [63, 145] show factors other than 2 for EB of the so-called external modes (defined as those propagating modes with either the smallest or the largest transversal momentum).

As we have found a marked anisotropy in the scattering in corrugated waveguides, and in order to quantify the EB, we define, for a given incoming mode $i$, an enhanced backscattering factor $\eta_i$ as

$$\eta_i = \frac{\text{Intensity in the backscattering mode}}{\text{Averaged intensity in the others}} = (N - 1) \frac{\langle R_{ii} \rangle}{\left( \sum_{j \neq i} \langle R_{ij} \rangle \right)}.$$  \hspace{1cm} (21)

Within the RMT framework all $\langle R_{ij} \rangle$ are equivalent and verify $2 \langle R_{ij} \rangle = \langle R_{ii} \rangle$ when time reversal symmetry is preserved ($\beta = 1$) and therefore

$$\eta_i = \frac{(N - 1)\langle R_{ii} \rangle}{\sum_{j \neq i} \langle R_{ii} \rangle / 2} = 2.$$ \hspace{1cm} (22)

Figure 12(c) shows $\eta_i$ versus $L$. While the ‘central’ modes ($i = 2, 3, 4$) present a factor $\eta \approx 2$, the ‘external’ ($i = 1, 5$) modes have a much larger enhanced backscattering factor ($\approx 4$).

The anisotropy of the scattering process, predicted in [63, 85, 86, 90], is especially relevant when the EB factor $\eta_i$ is addressed. In figure 13 we show the behaviour of $\eta$ as a function of the wavelength, for a length of the corrugated part such that the system is always in the
localization regime. The behaviour of the ‘external’ modes (defined as those propagating modes with either the smallest or the largest transversal momentum) is quite different to that of the ‘central’ ones. The lowest mode (i.e. that with the smallest transversal momentum) has a marked oscillatory value of $\eta_i$, showing peaks at the onset of each new propagating mode in the waveguide. For the rest of the modes, except for small oscillations, $\eta$ evolve from a high value just at the onset of propagation, to the expected factor of 2. This oscillating behaviour is closely related to that observed in the localization length $\xi$ shown in figures 13(a) and 6.

6. Statistics of transmitted waves

It is obvious that the knowledge of the average values alone do not provide a complete understanding of the transport properties. To do so, it is also necessary to have access to all the moments of the distribution; in other words the full distribution function of these coefficients is required. For the channel-to-channel coupling this becomes especially necessary in the localization regime, because here the fluctuation/mean-value ratio diverges exponentially [109]. However, when one is dealing with the field statistics it is in the transition and in the build up of the diffusion zone from the initially ballistic region where the distributions are not well known.

As shown above, the behaviour of the different averaged transmittivities for a surface disordered waveguide is slightly different from that obtained for bulk disordered wires. Therefore,
the question arising here is: does surface disorder change the probability distributions from those expected in bulk disordered systems?

In the case of random wires, using a combination of diagrammatic and RMT techniques [146, 147], it has been shown that, in the diffusive regime, the probability density \( P(T_{ij}) \) of the transmission channels is expected to behave according to a negative exponential (Rayleigh statistics) with tail deviations. However, the distribution \( P(T_i) \) within the same regime is Gaussian shaped but with tail deviations. Both distributions are expected [148] to evolve into the same lognormal distribution when going into the localization regime.

Using the same parameters as in the previous section, we depict in figure 14(a) the distribution \( P(T_i/\langle T_i \rangle) \) for different values of \( \langle T_i \rangle \). We observe that the distributions vary from mode to mode when they are extracted from samples with the same length. However, if we consider different modes with different lengths (always satisfying \( L \leq \xi \)) but having the same mean value the distribution is found to be the same. This means that, as long as \( L \leq \xi \), the distribution does not depend on the incoming mode \( i \) nor on the length \( L \), but only depends on the averaged transmission coefficients \( \langle T_i \rangle \), which act as a scaling parameter [106]. This assertion is clearly seen in the inset of figure 14(a) where \( P(T_i/\langle T_i \rangle) \) for \( i = 3, 4, 5 \) are plotted. All have the same average \( \langle T_i \rangle = 0.16 \), but correspond to different lengths of the disordered region (see figure 11). For the modes \( i = 1, 2 \) to have the same average the length required locates them into the localization regime and thus they have a different distribution. The speckle distribution \( P(T_{ij}/\langle T_{ij} \rangle) \) in the diffusive regime follows almost perfectly the Rayleigh statistics \( P(x) = \exp(-x) \) (see figure 14(b)). Deviations from this exponential distribution are observed when the length fulfills \( L \geq \xi \).
These points show that the dependence of \( \langle T_{ij} \rangle \) on length is not enough to fix the transport regime. While for \( \ell < L < \xi \) all coefficients are well described by the Rayleigh statistics, their dependence on \( L \) is completely different from one mode to the other, as observed in figures 11(a) and (b).

The distributions obtained for \( P(T_i/\langle T_i \rangle) \) and \( P(T_{ij}/\langle T_{ij} \rangle) \) exhibit the same qualitative behaviour as those obtained in the case of volume disordered wires [146–148] (compare figure 14 with figure 1 of [148] and figure 1 of [149]). It must be mentioned, however, that these conclusions do not hold if the length of the system is very small \( L \leq \ell \) (or the transmittivities are large \( \langle T_i \rangle \geq 1/2 \)).

Figure 15, where \( P(\ln(\tau)) \) is plotted versus \( \ln(\tau) \) (\( \tau = T_i, T_{ij} \)) for different values of \( \langle \ln(\tau) \rangle \), shows the aforementioned evolution of the distributions of both \( T_i \) and \( T_{ij} \) into the same lognormal distribution (solid lines):

\[
P(x) = \left( \frac{1}{\pi 2\sigma_x^2} \right)^{1/2} \exp \left[ -\frac{(x - \langle x \rangle)^2}{2\sigma_x^2} \right],
\]

with \( x = \ln(T_i), \ln(T_{ij}) \) and \( \text{var}(x) = \sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \).

When the length of the system is extremely large \( (L \gg \xi) \) we reach the so-called strong localization limit. In that case \( |\ln(\tau)| \approx L/\xi = L/(N\ell) \), and thus we obtain \( \sigma_x^2 \approx (3/2)|\langle x \rangle| \).

This means that the calculated distribution leads to a variance of 3/2, which is in contrast to the expected value of 2 [148]. However, the distribution obtained for the conductance in the localized regime is also a lognormal distribution and with a mean which is half the variance, in perfect agreement with RMT predictions [106, 109, 142] (see the inset of figure 15, where...
Figure 15. Distribution of $\ln(\tau)$ with $\tau = T_i, T_{ij}$ for different values of $\langle \ln(\tau) \rangle$. The inset shows the distribution of the conductance $G$ for two different lengths of the disordered region. Continuous lines are fit to lognormal distributions having $\sigma^2_{\langle x \rangle} = 3/2\langle x \rangle$, for $x = \ln(\tau)$, and $\sigma^2_{\langle x \rangle} = 2\langle x \rangle$ for $x = \ln(G)$. Reproduced with permission from [85]. © American Physical Society.

the solid lines show the corresponding lognormal distribution (equation 23 with $x = \ln(G)$) with $\text{var}(x) = \sigma^2_{\langle x \rangle} = 2\langle x \rangle$).

7. Statistics of reflected waves

From the analysis of the average values we have seen that the reflection coefficients are more sensitive than the transmission coefficients to the anisotropy of the scattering. The question that now arises is: will the scattering anisotropy affect the probability distributions?

Let us anticipate that the actual statistical distributions can still be described by the general RMT results (see Appendix A). The key point will be to consider the distributions as functions of the mean values rather than in terms of the number of channels:

$$P(R_{ii}, \beta = 1) = \left( \frac{1}{ \langle R_{ii} \rangle} - 1 \right) (1 - R_{ii})^{1/(\langle R_{ii} \rangle - 2}} \tag{24}$$

and

$$P(R_{ij}, \beta = 1) = \left( \frac{1}{ 2\langle R_{ij} \rangle} - 1 \right) (1 - R_{ij})^{1/(\langle R_{ij} \rangle - 3}} \times 2F_1\left( \frac{1}{ 2\langle R_{ij} \rangle} - 1, 1; \frac{1}{ \langle R_{ij} \rangle} - 2, 1 - R_{ij} \right), (i \neq j). \tag{25}$$

When time reversal symmetry is removed, the statistical law coincides with that obtained for $R_{ij}$ in the case of $\beta = 1$, i.e.

$$P(R_{ij}, \beta = 2) = P(R_{ii}, \beta = 2) = P(R_{ii}, \beta = 1). \tag{26}$$
7.1 Distribution of the modes in the transition from diffusive to localization regimes

In figure 16 we have plotted equations (24) (left) and (25) (right) for a set of values of $\langle R_{ii} \rangle$ and $\langle R_{ij} \rangle$ respectively, together with the numerical results for a surface corrugated waveguide. The geometrical parameters used in this case are exactly the same as those used for the transmission case and the distributions have been obtained over an ensemble of 1000 independent configurations.

The numerical probability densities have been calculated in the range of wavelengths shown in figure 13, for different length values including both the diffusive ($\xi/N \lesssim L \lesssim \xi$) and localized ($L \gtrsim \xi$) regimes. The histograms in figure 16 show the intensity distributions obtained numerically for different averages of either $\langle R_{ii} \rangle$ or $\langle R_{ij} \rangle$. We would like to emphasize that those distributions corresponding to a different number of propagating modes and a different transport regime, but having the same average of the reflection coefficient, are indistinguishable. Despite the scattering being highly non-isotropic, it is remarkable that both numerical and analytical results coincide with each other, without any adjustable parameter.

This means that the statistical law governing the backscattered intensities $R_{ii}$ does not depend on the time reversal symmetry conditions, as long as they are expressed in terms of the mean value as a scaling parameter.

When $N \gg 1$ (i.e. when the mean values are very small) equations (24) and (25) approach the well known negative exponential (Rayleigh) distribution

$$P(R_{ij}) = \frac{1}{\langle R_{ij} \rangle} \exp \left( -\frac{R_{ij}}{\langle R_{ij} \rangle} \right). \quad (27)$$

In the opposite limit of small $N$, coherent backscattering effects manifest themselves in the distributions. This is illustrated in the analysis of the case $N = 2$ ($\langle R_{ii} \rangle = 2/3$ and

![Image](image_url)

Figure 16. Left: distributions of $R_{ii}$ for different values of $\langle R_{ii} \rangle$. Thin solid lines correspond to the numerical calculations for $\langle R_{ii} \rangle \sim 0.15, 0.25, 0.35, 0.45$ and $0.51$, respectively, whereas thick solid lines are the analytical predictions. Right: distributions of $R_{ij} (i \neq j)$ for different values of $\langle R_{ij} \rangle$. Thick solid lines are the analytical predictions for the same values of the averages as those indicated for the numerical histograms (thin solid lines) $\langle R_{ij} \rangle \sim 0.05, 0.10, 0.15$ and $0.2$ respectively. Reproduced with permission from [87], © American Physical Society.
Figure 17. Probability density for $N = 2$: $P(R_{ii})$ ($\langle R_{ii} \rangle = 2/3$) and $P(R_{ij})$ ($\langle R_{ij} \rangle = 1/3$). Thick continuous lines are the RMT predictions and thin broken lines the numerical results. Reproduced with permission from [87], © American Physical Society.

$\langle R_{ij} \rangle = 1/3$ where the probabilities for $i = j$ and for $i \neq j$ are quite different (see figure 17 for a comparison with the numerical results):

$$P(R_{ii}, \beta = 1) = \frac{1}{2\sqrt{1 - R_{ii}}}$$  \hspace{1cm} (28)

and

$$P(R_{ij}, \beta = 1) = \frac{1}{2\sqrt{R_{ij}}}, \quad (i \neq j).$$  \hspace{1cm} (29)

### 7.2 Intensity distributions

One could suggest that this behaviour lies in the fact that we are considering the mode or angular distribution. However, it is also possible to obtain the intensity distribution both numerically and, for small $N$, analytically within the RMT framework. Now consider the case of single mode excitation for a two-mode waveguide ($W_0/\lambda = 1.4$, this means $\ell/W_0 \approx 52$ and $\xi \approx 2\ell$). Let us analyse the distribution of the reflected intensity *at a given point* $P(I(r))$, where $I(r) = |E(r)|^2$. In terms of the joint probability distribution of the elements of the reflection matrix $P(\{r_{nm}\}; \beta)$(see Appendix A) it is given by:

$$P_n(I(r), \beta) = \int P(\{r_{nm}\}; \beta)\delta(I - |E_n(r)|^2) \prod_{m=1}^{N} dr_{nm}.$$  \hspace{1cm} (30)

The reflected electromagnetic field for a single incoming mode $n$ at any given point $r = (x, z)$ outside the disordered region $E_n(r)$ is written in terms of the scattering matrix
elements as:

\[ E_n(x, z) = \sum_m \frac{1}{\sqrt{K_m}} r_{nm} \Phi_m(x) \exp(-i K_m z) \]

\[ \approx \sum_m |e_{nm}(x)| r_{nm} \exp(-i K_m z), \tag{31} \]

where \( \Phi_m(x) \) are the normalized transversal eigenfunctions \( (\Phi_m(0) = \Phi_m(W_0) = 0) \) with momenta \( q_m = m\pi/W_0 \) and where \( K_m^2 = (2\pi/\lambda)^2 - q_m^2 \).

In figure 18 we have plotted the numerical probability densities (filled circles correspond to an incoming mode \( n = 1 \) and open circles to \( n = 2 \)) for the backscattered intensity in our 2D disordered waveguide at \( x = 0.56W_0 \). To avoid the influence of any specular contribution to the backreflected field we have subtracted its mean value \([91, 113]\), thus the scattered intensity at a point is \( I(r) = |E(r) - \langle E(r) \rangle|^2 \). The calculations have been made for a system length \( L \) such that the system is in the localization regime \((L \approx 6\xi)\) and with the phase of the reflected field completely randomized (see section 8).

In the RMT context closed expressions for \( P_n(\bar{I}, \beta = 1) \) \((\bar{I} = I/I_{\text{max}})\) can be obtained for small values of \( N \) only, but it is very difficult to work them out for arbitrary \( N \) (when time reversal symmetry is broken, simple formulas are given for any \( N \)). In the simplest case, in which only two propagating modes are allowed, equation (30) can be integrated \([150]\) to give:

\[ P_n(\bar{I}, \beta = 1) \propto \frac{1}{\chi_+} F \left( \frac{\pi}{2}, \sqrt{1 - \frac{\chi_-^2}{\chi_+^2}} \right), \tag{32} \]

where \( F(\pi/2, x) \) is the complete elliptic integral of the first kind \([151]\), \( \chi_{\pm} \) is given by

\[ \chi_{\pm}^2 \equiv \bar{I}(A_{n2} - A_{n1}) + A_{n1} \pm 2\sqrt{A_{n1}A_{n2} \bar{I}(1 - \bar{I})} \tag{33} \]

and \( A_{nm} \) are the normalized intensities in each mode, \( A_{nm}(x) = |e_{nm}(x)|^2/I_{\text{max}} \). The distributions present a functional dependence which strongly differs from those obtained in the \( \beta = 2 \) case. The intensity distributions present clear singularities at \( \bar{I} = A_{nm}(x) \) as well as an explicit modal anisotropy due to their dependence on the momenta, and are highly dependent on the spatial position through \( A_{nm}(x) \).

Lines in figure 18 correspond to the theoretical predictions for \( n = 1 \) and \( n = 2 \) with no fitting parameter, showing a remarkable agreement between the theoretical predictions and the numerical results.

The physical origin of the peaks in the intensity distribution can be understood in terms of the distribution of the reflection coefficients. We have shown above that the distribution \( P(R_{nn}) \), for any \( N \), is given by

\[ P(R_{nn}) = \left( \frac{1}{\langle R_{nn} \rangle} - 1 \right) (1 - R_{nn})^{(1/(R_{\text{max}}) - 2)}. \tag{34} \]

If the average intensity in the backscattering direction is larger than 1/2 \((R_{nn} > 1/2)\), this distribution diverges at \( R_{nn} = 1 \), i.e. the most probable case corresponds to total reflection through the backscattering channel. In contrast, the distributions for \( R_{nm} \) \((n \neq m)\) always have a maximum at \( R_{nn} = 0 \). For example, for \( N = 2 \) we have (equations (28) and (29),

\[ P(R_{12}) = 1/(2\sqrt{R_{12}}) \] and \[ P(R_{11}) = 1/(2\sqrt{1 - R_{11}}) \].

These distributions diverge at the most probable values \( R_{12} = 0 \) and \( R_{11} = 1 \). The origin of the strong peak in \( P_n(\bar{I}, \beta = 1) \) is then associated with the most probable case of total intensity reflected in the backscattering mode \((R_{11} = 1, R_{12} = 0)\), i.e. \( \bar{I} = A_{nn}(x) \).

From this argument we expect that, for arbitrary \( N \), the distribution of the spatial intensity strongly depends on the enhanced backscattering factor (equation (21)). When \( (R_{nn}) < 1/2 \) the
Wavetransport through waveguides

Figure 18. Numerical distributions for a 2D surface disordered waveguide for $x = 0.56W_0$ and $N = 2$. Filled circles correspond to the incidence through the mode $n = 1$, open circles correspond to $n = 2$, and lines correspond to the RMT theoretical predictions. Reproduced with permission from [150], © American Physical Society.

distributions should present a peak at $\tilde{I} = 0$ and evolve continuously to a Rayleigh distribution at $N \to \infty$. In contrast, when $\langle R_{nn} \rangle > 1/2$, the distribution should present an anomalous peak at $\tilde{I} = A_n(x)$ corresponding to $R_{nm} = \delta_{nm}$.

In order to check this behaviour we have calculated the probability densities in our 2D disordered waveguide for the same parameters as before, but with $W_0/\lambda = 2.6$, which corresponds to $N = 5$. In figure 19 we show the results for the incoming mode $n = 3$ with $\langle R_{33} \rangle = 0.34$.

Figure 19. Numerical distributions for a 2D surface disordered waveguide for $x = 0.56W_0$ and $N = 5$. Squares correspond to the incidence through the mode $n = 3$ ($\langle R_{33} \rangle = 0.34 < 0.5$) and circles to $n = 1$ ($\langle R_{11} \rangle = 0.51 > 0.5$). Reproduced with permission from [150], © American Physical Society.
(squares) and for \( n = 1 \) with \( \langle R_{11} \rangle = 0.51 \) (circles). For \( n = 3 \) the distribution peaks at \( \tilde{I} = 0 \) while for \( n = 1 \) it peaks at \( \tilde{I} \approx A_{11} \) as expected from the previous argument.

8. Transition between ballistic and diffusive regimes

The preceding sections have been mainly concentrated on the probability densities and statistical momenta of the different transport coefficients (or, in other words, the intensity of wavefunction angular spectrum), both in the diffusion and in the localization regimes. The statistics of the spatial intensity in quasi-one-dimensional systems have been studied [152] assuming that the point source and detector are both embedded in the bulk of the medium. However, the field statistics in the transition and in the build up of the diffusion zone from the initially ballistic region are not well known. In one-dimensional systems with weak disorder, theoretical results [153, 154] showed the existence of a phase randomization length even though phase randomness is not present at each scattering event. This phase randomness for each scattering event is required in order to obtain the Rayleigh distribution, which involves a complete randomization of the phase. However the situation in more dimensions has not been studied in detail.

In recent experimental work [113] it has been possible to measure the probability distribution of a microwave field transmitted through a waveguide with volume disorder in the cross-over from ballistic to diffusive regimes (see figure 20). Under these conditions the electromagnetic

![Figure 20](image)

Figure 20. Distribution of normalized amplitude and phase of the total field (squares) and the residual field (circles) for two waveguides of different lengths. Reproduced with permission from [113], © American Physical Society.
field arriving at the observation point had a considerable portion that suffered no scattering (coherent or ballistic component). The distributions of amplitudes of both the total and the residual field, measured in the near field speckle pattern of the transmitted wave, were found to be in remarkably good agreement with predictions based on a phenomenological model. This model considers the wavefield as a sum of a circular random phasor and a constant that takes into account the coherent component (random phasor sum, RPS) [155] and results derived from diagrammatic techniques [156]. However, while the phase distributions of the total field were in good agreement with the theoretical predictions [155], clear deviations from the RPS predictions were expected for single-mode excitation [113]. The aim of this section is to perform a detailed analysis of the statistics associated with the single-mode excitation. As we will see, the statistics arising for surface disorder present strong similarities with that obtained for volume disorder [91].

Following the preceding sections, the geometrical parameters of our corrugated waveguide will be $W_0/\delta = 7, l/\delta = 3/2$ and $W_0/\lambda = 2.6$, with $\lambda$ being the wavelength. Conductance calculations (see figures 4 and 6) for this particular choice of waveguide geometry (which allows five propagating modes, i.e. $N = 5$) give a transport mean free path $\ell/W_0 \approx 6.5$ and a localization length $\xi \approx N\ell$. For a single incoming mode $n$, the transmitted field $E_n(r)$ at any given point $r$ beyond the disordered region is given by:

$$E_n(x, z) = \sum_m t_{nm} \frac{\Phi_m(x)}{\sqrt{K_m}} \exp(i K_m z), \quad (35)$$

where $\Phi_m(x)$ are the normalized transversal eigenfunctions with momentum $q_m = m\pi/W_0$; $K_m^2 = (2\pi/\lambda)^2 - q_m^2$ and $t_{nm}$ are complex amplitudes. The complex field, as a random function of position, can be written as the sum of the average (coherent) and residual (diffuse) fields, $E(r) = \langle E(r) \rangle + \delta E(r)$, where the $\langle \rangle$ represent the ensemble average over different disorder realizations. The coherent and residual intensities can be written as $I_c = |\langle E \rangle|^2$ and $I_{res} = |\delta E|^2$, with the average total intensity $\langle I \rangle = \langle I_{res} \rangle + I_c$. In the following section we will present results for the speckle contrast (SC) involving $5 \times 10^3$ independent realizations, as well as for the total distributions where the ensemble is $2 \times 10^5$.

### 8.1 Random phasor sum

When the field arriving at the observation point has a measurable coherent or ballistic part, the negative exponential distribution is no longer valid. To account for the new effects one can add a constant to the diffuse (scattered) components of the field and thus the joint probability of the real and the imaginary parts of the electromagnetic field is:

$$P(A_r, A_i) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{(A_r + \sqrt{I_c})^2 + A_i^2}{2\sigma^2} \right\}, \quad (36)$$

where $2\sigma^2 = |\langle |E|^2 \rangle - |\langle E \rangle|^2| = \langle I \rangle - I_c$.

Then the statistics of amplitude $A = E/\sigma$ and phase $\phi$ (see equations (2.59) and (2.68) in [155]) are:

- **Amplitude distribution**:

$$P(A) = A \exp \left( -\frac{A^2 + \gamma^2}{2} \right) I_0(\gamma A), \quad (37)$$

where $I_0(x)$ is the modified Bessel function of the first kind of zero order.
Phase distribution:

\[ P(\phi) = \frac{1}{2\pi} \exp\left(-\frac{\gamma^2}{2}\right) + \frac{\gamma \cos \phi}{\sqrt{2\pi}} \exp\left(-\frac{(\gamma \sin \phi)^2}{2}\right) \Phi(\gamma \cos \phi), \]  

(38)

where

\[ \gamma^2 = \frac{2I_c}{\langle I \rangle - I_c} \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \text{erf} \frac{y^2}{2} dy. \]

8.2 The speckle contrast in a surface disordered waveguide

Let us first focus on the behaviour of the speckle contrast \( SC(I) = \sigma_I/\langle I \rangle \) (with \( \sigma_I^2 \equiv \text{var}(I) \)) as a function of the length \( L \) of the disordered part of the waveguide. We consider \( SC \) for both the total intensity (\( SC(I) \)) and for the residual diffuse intensity (\( SC(I_{res}) \)). As seen in figure 21, the behaviour of these two quantities highly depends on the incoming mode \( n \). Except for the small oscillations at short lengths, \( SC(I) \) increases smoothly as a function of \( L \). These small oscillations are due to multiple internal reflections at the boundary of the disordered region. Once the wave has traversed a certain length, which strongly depends on \( n \), the coherent intensity becomes negligible (\( I_c/\langle I \rangle \ll 1 \) and \( SC(I) \approx SC(I_{res}) \)). Here again we find the highly anisotropic transport characteristics of surface randomly corrugated waveguides. For high order modes (\( n > 2 \)) there is almost no difference between total and residual intensities even for very short lengths, reflecting the fact that for high order modes a small amount of scattering is enough to wash out the coherent component. For these modes \( SC \) changes rapidly as a function of \( L \), reflecting non-Rayleigh statistics all the way from the ballistic to the localization regime. As a matter of fact, it is difficult to identify a clear diffusive (Rayleigh) regime for these modes.

Figure 21. Speckle contrast \( \sigma_I/\langle I \rangle \) as a function of the length of the disordered part for all the propagating modes. Symbols represent the speckle contrast for the residual intensity and lines represent the speckle contrast for the total intensity. Reproduced with permission from [91], © American Physical Society.
SC for the residual diffuse intensity shows interesting behaviour as a function of \( L \). For the lowest order mode \( n = 1 \), and for very short lengths (i.e. a very small number of scattering events), SC is lower than 1. As the length increases, there is a maximum (SC > 1) followed by a slow decrease of SC towards its Gaussian value (SC = 1). This behaviour of SC(\( I_{\text{res}} \)) is fully consistent with that expected from non-Gaussian statistics of a system with a small number of scatterers [157–159]. However, as \( L \) approaches the localization length, SC starts to increase continuously. This non-Gaussian speckle now has a different physical origin, it arises as a consequence of multiple scattering effects that can also be found, for example, when speckle patterns propagate through the atmosphere or from a cascade of multiple diffusors (‘speckled speckle’ statistics [160]). Finally, in the asymptotic limit \( L/\xi \gg 1 \), the intensity at a point results from a large number of scattering processes along the waveguide and, therefore, one expects the logarithm of this intensity to be a Gaussian random variable [106]. This is shown in figure 22(a), where we plot the distributions for \( \ln(I) \) (incoming modes 2 and 5) deep in the localization regime. We find that, contrary to the transmission case, the variance is not constant but depends on the incoming mode, \( \sigma_I/\langle I \rangle \propto (L/\xi)^{1/4} \exp(L/4\alpha_n\xi) \). (39)

Then, in the strong localization regime, SC increases exponentially with length (\( \ln(SC) \propto L/\xi \)).

### 8.3 Field and phase distributions in ballistic wave propagation

Open and filled circles in figure 23 show the distribution of the normalized total and residual field amplitudes respectively \( A = E/\sigma \) (being \( 2\sigma^2 = \langle |E|^2 \rangle - |\langle E \rangle|^2 \)) (figure 23(a)), and phase (figure 23(b)) at a point outside the disordered region of the waveguide when its length is \( L/W_0 = 3.6 \) (incoming mode = 2). In this case we have a non-negligible coherent component indicating that the transport has a ballistic contribution. As SC = 1 for the diffuse field, we can compare with the RPS model for both the total and the residual fields. In this model

![Figure 22. (a) Distributions for the logarithm of the intensity (incoming modes 2 and 5) in the localization regime (\( L \approx 13\xi \)). The distributions are lognormal with variance depending on the incoming mode. (b) Speckle contrast versus \( L/W_0 \) for the same incoming modes as in (a). Thin continuous lines are the theoretical curves assuming a lognormal distribution.](image-url)
the residual field amplitude follows the Rayleigh distribution, while the phase is uniformly distributed. The total field amplitude and phase distributions are described by a single scaling parameter $I_c/\langle I \rangle$ which can be obtained directly from the numerical calculation. Continuous lines in figure 23 are the theoretical curves obtained directly from equations (37) and (38). As shown in figure 23, the distributions for both amplitudes as well as that for the phase of the total field show very good agreement with the numerical calculations.

It is worth noting that the phase distribution of the residual field for short lengths (filled circles in figure 23(b)) is far from uniform, contrary to that expected from the RPS model (solid straight line in figure 23(b)). The similarity between our calculations and the experimental results shown in figure 20 (figure 3 in [113]) is remarkable. This non-uniform distribution reflects a very small number of scattering events along the waveguide.

### 8.4 Field and phase distributions in diffusive wave propagation

When the speckle contrast is non-Gaussian, the RPS results for the field amplitude can no longer be applied. In this case, we find that the distribution of amplitudes of the residual field is fixed by the speckle contrast and is well reproduced by the speckled-speckle distributions...
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Figure 24. Same as figure 23 but for the case of SC > 1 (incoming mode \( n = 3, L/W_0 = 18.1 \)). (a) The solid line corresponds to a ‘\( K \)’ distribution while the dashed line corresponds to the Rayleigh distribution. Note that the total and residual fields are superimposed indicating that the ballistic contribution to the field amplitude is almost negligible. (b) Phase distribution of the total and residual fields. The phase of the residual field is now uniformly distributed, whereas that of the total field is not uniform yet, reflecting that the coherent part is negligible but not zero. Reproduced with permission from [91], © American Physical Society.

(so-called ‘\( K \)’ distributions [157–160]):

\[
P(A) = \frac{2\sqrt{2M}}{\Gamma(M)} \left( \frac{MA^2}{2} \right)^{\frac{M}{2}} K_{M-1}(A\sqrt{2M}),
\]

where \( K_{\nu}(x) \) is the modified Bessel function of the second kind of order \( \nu \) and \( 2/M = SC(I_{\text{res}})^2 - 1 \). When \( SC = 1 \), the Rayleigh distribution is recovered. Figure 24(a) shows the amplitude distribution for \( L/W_0 = 18.1 (n = 3) \). The distributions for both total and residual fields appear superimposed, reflecting how the coherent intensity becomes negligible when increasing the length of the disordered section of the waveguide. The solid line in figure 24(a) indicates that the ‘\( K \)’ distribution follows the numerical data remarkably well. As a general result, we found that the distribution of amplitudes of the residual field can be well described by a ‘\( K \)’ distribution, characterized by a single scaling parameter \( SC(I_{\text{res}}) \), all the way from ballistic to near localization regime (i.e. within our length window). Notice that values of \( SC \) greater than one can occur even when the transport is almost ballistic as discussed above for \( n = 1 \) (figure 21). Even in this case, the ‘\( K \)’ distributions give a good description of the
amplitude statistics, although the physical origin here is associated with the small number of scatterers [157–159].

Phase randomization increases with the number of scattering events (i.e. as the length increases or, for a given length \( L \), as the order of the incoming mode increases). This is clearly visible in figure 24(b) where the phase distributions for \( L/W_0 = 18.1 \) \((n = 3)\) are shown. The phase of the residual field is now uniformly distributed and that of the total field still follows the RPS prediction. The agreement of the phase distributions with RPS is remarkable even though the speckle contrast is clearly non-Gaussian \((SC > 1)\). Notice that the phase distribution of the total field is extremely sensitive to the coherent non-diffuse field. Even for \( I_c/I < 0.01\), where there is almost no difference between total and diffuse amplitude distributions, the phase distribution of the total field is still far from randomized.

9. Conductance distributions

This section is devoted to the analysis of the distribution of the conductance. The description of the precise shape of this distribution is among the most desired goals in the physics of the last thirty years.

It has been obtained in different limiting cases such as in the pure diffusive and deep localization regimes in the large \( N (N \gg 1) \) and within different theoretical approaches (RMT and diagrammatic expansions) and tested by numerical techniques for bulk disordered wires [5, 8, 109, 142]. The results are that the distribution of the conductance \( P(G) \) is Gaussian shaped in the diffusive regime, whereas the conductance exhibits a lognormal distribution in the localization regime, i.e. exactly the same behaviour as for the transmission coefficients. For rough wires, we are more concerned about the small-\( N \) limit. Even then, calculations made in the deep localization regime show this behaviour is shown to be valid (see inset in figure 15).

A more intriguing question arises when the distribution of the conductance is to be described in the so-called transition region (the region in which the transition from the diffusive to the localization regime takes place). In this region, the distribution of the conductances are difficult to explore. The first results came from studies related to the conductance distribution at an integer quantum hall transition [161–165] where striking results for the shape of \( P(G) \) were reported. More recently, and based on the Dorokhov–Mello–Pereyra–Kumar (DMPK) equation [107, 110] for quasi-1D wires in the absence of time reversal symmetry it has been found [166] that, the cross-over region between metallic and insulating regimes is highly non-trivial, and shows one-sided log normal distributions for the conductance at the transition.

In order to shed light on this matter, we present calculations for three different values of \( W_0/\lambda \): 4.9, 2.6 and 1.8, allowing \( N = 9, 5 \) and 3 propagating modes respectively. We have always considered \( l/\delta = 3/2\), but with two different ratios \( W_0/\delta = 7 \) (for \( N = 5, 3 \)) and \( W_0/\delta = 13.25 \) (for \( N = 9 \)).

The analysis of the conductance distributions \( P(G) \) along the transition region (see figure 25) shows that they evolve far from smoothly from a Gaussian-shaped distribution to the lognormal distribution as one could expect from the behaviour of the transmission coefficients. In figure 25 we show \( P(G) \) for some selected values of \( \langle G \rangle \) \((\langle G \rangle \approx 1 \) (a), \( 4/5 \) (b), \( 1/2 \) (c) and \( 1/3 \) (d)). The results obtained for \( N = 9 \) \((\circ)\), \( N = 5 \) \((\triangle)\) and \( N = 3 \) \((\blacksquare)\) are almost identical (within our numerical accuracy) suggesting that the only parameter determining the exact shape of \( P(G) \) is the average conductance \( \langle G \rangle \) as expected from scaling arguments. This result only holds for \( \langle G \rangle < 1 \). For \( \langle G \rangle \) larger than 1, i.e. in the diffusive regime, the results of the three waveguides are different; while for \( N = 9 \) the distribution is close to a Gaussian, for small \( N \)
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Figure 25. Conductance distributions for different averaged conductance values: (a)\(\langle G \rangle = 1\); (b)\(\langle G \rangle = 4/5\); (c)\(\langle G \rangle = 1/2\); (d)\(\langle G \rangle = 1/3\). Symbols are for \(N = 9 (\diamond)\), \(N = 5 (\circ)\) and \(N = 3 (\blacksquare)\). Continuous lines represent the analytical results of the random matrix approach. Reproduced with permission from [94], © American Physical Society.

\(P(G)\) presents strong finite size effects (A. García-Martín and J.J. Sáenz, unpublished data). It is remarkable that the almost perfectly flat distribution obtained for \(\langle G \rangle = 1/2\) (figure 25(c)) is in full agreement with those obtained for 2D systems in the quantum Hall regime at the critical point [164, 165]: a flat distribution except for a small dip at \(G \approx 0\) and an exponential cutoff for \(G > 1\).

As the averaged conductance goes slightly below 1/2, \(P(G)\) becomes a ‘one-sided’ log normal distribution with a sharp cut off at \(\langle G \rangle > 1\) (see figure 26) in agreement with both numerical [162, 163] and analytical [166] results.

In figure 26 we have plotted our results for \(P(\ln G)\) for several values of \(\langle G \rangle\) and also the best fit to a one-sided log normal distribution (thick solid lines) for each one. Despite the very good agreement, provided \(\langle G \rangle < 1/2\), the distribution at the critical value of 1/2 is clearly described by \(P(\ln G, \langle G \rangle = 1/2) = \exp(\ln G)\) (thin line in figure 26(a)) which corresponds to a flat distribution for \(P(G, \langle G \rangle = 1/2) = 1\), \(0 \leq G \leq 1\).

In order to get deeper insight into these surprising results the aid of different quantities is required. The most suitable ones are the transmission eigenchannels \(\{\tau_i\}\). We have already seen before that the eigenchannels are defined as the eigenvectors of the matrix \(tt^\dagger\) and that they might be seen as the most natural basis to analyse the properties of the conductance \(G = \text{trace}(tt^\dagger) = \sum_i \tau_i\).

In figure 27 we show the evolution of the average value of the eigenchannels as a function of \(\langle G \rangle\). In the diffusive regime \((\langle G \rangle > 1)\), in agreement with the behaviour discussed by Imry [106], almost all the eigenchannels are either fully open or closed and only a few of them fluctuate giving rise to universal conductance fluctuations UCF. However, their exact behaviour depends on the particular parameters of the wire. Close to the cross-over to localization, when all the eigenchannels are closed and only two (or one in the localization threshold) of them are partially open, the average eigenchannel transmissions (figure 27) as well as the conductance distributions (figure 25) do not depend on the size of the wire or the defect details. In this
sense, these results suggest a universal behaviour of the conductance with \( \langle G \rangle \) as the only scaling parameter.

Recent results for bulk disordered systems do confirm this general result [167]. In fact, the conductance distributions for \( G < 1/3 \) are indistinguishable. However, the detailed shape of the distribution in the transition region (\( 1/3 \lesssim G \lesssim 1 \)) and, especially the singular ‘cusp’

Figure 26. Distribution of \( \ln G \) for different values of the average conductance beyond the critical point (\( \langle G_c \rangle < 1/2 \)). Thick solid lines are the best fits to a one-sided log normal distribution. The thin solid line in (the top left panel) corresponds to \( P(\ln G, \langle G \rangle = 1/2) = \exp(\ln G) \) (i.e. \( P(G, \langle G \rangle = 1/2) \) is a uniform distribution). Reproduced with permission from [94], © American Physical Society.

Figure 27. Averaged transmission eigenvalues \( \langle \tau_i \rangle \) as a function of the averaged conductance. Dashed line corresponds to a 9-mode wire, solid line to a 5-mode wire and long-dashed line to a 3-mode wire. Reproduced with permission from [94], © American Physical Society.
points [94] observed in surface disordered waveguides are of a different nature to those obtained for bulk disorder. The origin of this difference is still a matter of controversy.

In contrast with the insulating regime \( \langle G \rangle < 1/2 \), where the conductance is known to be well described by one-sided log normal distributions [166], in the cross-over regime \( \langle G \rangle > 1/2 \) where the statistics are dominated by one or two fluctuating channels, there is no analytical result available for the conductance distribution, except for in the vicinity of \( G = 1 \) where the distribution shows non-analytical behaviour [167–169]. As a first analytical approach to this problem, given the reduced knowledge of the statistical correlations between different eigenchannels, a possible choice of the statistical ensemble is that which maximizes the information entropy subject to the known constrains of flux conservation and time-reversal invariance. In the RMT context, this leads to the circular orthogonal ensemble. From figure 27 we also know that, close to the onset of localization (for \( 1/2 < \langle G \rangle < 1 \)), only two eigenchannels actually contribute to the conductance, independently of the initial number of channels \( N \). From the polar decomposition of the scattering matrix [109, 170, 171] it is possible to write the joint probability distribution of \( n \) transmission eigenchannels (with \( \tau_i = 0, i = n + 1, \ldots, N \)) as

\[
P(\{\tau_i\}) \propto \prod_{i<j} |\tau_i - \tau_j| \prod_{k} \tau_k^{z_k}
\]

where \( \alpha = (1+n)(\langle G \rangle - n/2)/(n-\langle G \rangle) \). This expression is similar to that obtained for chaotic cavities (see equations [76], [93] in [109]). In another context, equation (41) would correspond to the exact RMT distribution for randomly coupled wide–narrow–wide leads [108].

Within this approach, the conductance distribution is fully described by the number of fluctuating channels \( n \) and the mean value of the conductance \( \langle G \rangle \), which, in terms of the number of channels \( N \), is given by \( \langle G \rangle = n(2N - n)/(2N + 1) \). Simple closed expressions for \( P(G; \langle G \rangle, n) \) can be obtained for \( n = 1, 2 \),

\[
P(G, \langle G \rangle, 1) \propto G^{1-2\langle G \rangle} \quad \text{for} \quad 0 < G < 1
\]

and

\[
P(G, \langle G \rangle, 2) \propto \begin{cases} 
\left( \frac{G}{2} \right)^{2(1-2\langle G \rangle)/(n-2)}, & \text{if} \quad 0 < G \leq 1 \\
\left( \frac{G}{2} \right)^{2(1-2\langle G \rangle)/(n-2)} - (G - 1)^{1-2\langle G \rangle}, & \text{if} \quad 1 \leq G < 2.
\end{cases}
\]

In figure 25 we have plotted the RMT distributions (continuous thick lines) for different \( \langle G \rangle \) (\( n = 1 \) for \( \langle G \rangle \leq 1/2 \) and \( n = 2 \) for \( 1/2 \leq \langle G \rangle \)), together with our numerical results. As can be seen, there is very good agreement between the analytical results and the numerical calculations. In particular, our RMT approach captures some important features of the distributions such as the almost perfectly flat distribution at \( \langle G \rangle = 1/2 \) and the ‘cusp’ point in \( P(G, \langle G \rangle) \) at \( G = 1 \).

10. Summary

We have carried out an exhaustive and systematic analysis of classical (scalar) wave propagation through surface-disordered waveguides. We have put forward some analogies between electron transport in wires and classical wave propagation through surface disordered waveguides, and thus defined transport regimes in analogy to those for electron transport. One
important observation has been that both mean free path and localization length oscillate as functions of the wavelength, with minima close to the onset of a new mode. These oscillations have been related to those observed whenever the surface presents a single defect and can be associated with resonant scattering induced by quasi-bound states.

A special feature of classical wave propagation is the capability of single-mode excitation. This has allowed the study of the statistical properties of transmission and reflection coefficients (transmitted and reflected speckle pattern respectively). It has been shown that in both cases the averages exhibit considerable anisotropy, with preference to forward coupling through the lowest mode in the transmission case. In the reflection case the anisotropy has induced strong backscattering effects, which have shown a marked dependence on the wavelength, with oscillations closely related to those observed for both mean free path and localization length.

The numerically obtained full distribution functions of the transmission and reflection coefficients have been compared with theoretical and experimental results, resulting in a considerably good agreement. As a general conclusion, we have put forward that, although the mean values can deviate considerably from the RMT results, the fluctuations are perfectly described. Also, when the number of modes is small, we have addressed the completely anomalous behaviour for the distribution of the reflection coefficients that, given the independence on the scattering details, should be observable in different systems under rather general conditions. This point has been shown to be especially relevant in the case of the spatial intensity distributions.

The transition from the ballistic to the diffusive regime has been addressed by analysing the behaviour of amplitude and phase of both the coherent (unscattered) and the diffuse fields. The numerical distributions have been compared with previous experimental results as well as with a phenomenological model, and showed a remarkable agreement. Also, the observed deviations from Rayleigh’s statistics in the diffusive regime have been found to be consistent with the so-called ‘K’ distributions.

In the transition from the diffusive to the localization regime, we have found that close to the cross-over the distributions of the conductance present unexpected shapes. The shapes have been found to be independent of the details of the system with the averaged conductance $\langle G \rangle$ as the only scaling parameter. It has been shown that for $\langle G \rangle$ between 1/2 and 1, the statistics are dominated by one or two fluctuating eigenchannels and the numerical conductance distributions are surprisingly well described by RMT results. At a critical value $\langle G \rangle_c = 1/2$, the distribution has turned out to be almost perfectly flat. In the insulating regime the conductance distributions for surface disordered systems for $G < 1/3$ have proved to be indistinguishable from those corresponding to bulk disorder. However, the detailed shape of the distribution in the transition region ($1/3 \lesssim G \lesssim 1$) has exhibited remarkable discrepancies whose nature is different from that obtained for bulk disorder.

Acknowledgements

We have benefited from discussions with A. Cano, R. Carminati, G. Cwilich, V. Freylikher, L.S. Froufe-Pérez, P. García-Mochales, R. Gómez-Medina, J.J. Greffet, M. Governale, T. López-Ciudad, P.A. Mello, A.D. Mirlin, K.A. Muttalib, A. Rosch, G.M. Sacha, J.A. Sanchez-Gil, F. Scheffold, P.A. Serena, J.A. Torres and P. Wölfle. We are especially grateful to M. Nieto-Vesperinas for valuable and continuous support and discussions. J.J.S. acknowledges the hospitality offered to him by the EMC2 Laboratoire during his sabbatical at Ecole Centrale Paris (supported by the Spanish MECyD, Ref. No. PR2002-0161). A.G.-M. acknowledges support from the Spanish MECyD through the programme Ramón y Cajal.
Appendix A. Intensity distributions in random matrix theory

Let us consider a scattering matrix with its elements as random as possible, i.e. the only correlations between elements come from the unitarity and reciprocity conditions. The joint probability distribution of the $2N$ elements in a given row $i$ of a $2N \times 2N$ random scattering matrix was derived by Pereyra and Mello [172] in the presence ($\beta = 1$) or absence ($\beta = 2$) of time reversal invariance:

$$P(s_{1i}, s_{2i}, \ldots, s_{i(2N)}) \propto (1 - |s_{ii}|^2)^{-\delta_{\beta_1}} \frac{2i-1}{2} \delta \left(1 - \sum_m |s_{im}|^2\right). \quad (A1)$$

Note that $P([s_{ij}])$ actually depends only on the squared moduli of the complex matrix elements. This joint probability distribution is expected to describe some general characteristics of chaotic cavities where there is no difference between transmitted and reflected waves (except the well known enhanced backscattering factor).

It was pointed out [108] that the statistics of reflection coefficients in the localization regime (where the transmission amplitudes go to zero) could be described by a similar distribution for the $N \times N$ elements of the random scattering (reflection) matrix $r$:

$$P(r_{i1}, r_{i2}, \ldots, r_{iN}) \propto (1 - |r_{ii}|^2)^{-\delta_{\beta_1}} \frac{N-1}{2} \delta \left(1 - \sum_m |r_{im}|^2\right). \quad (A2)$$

The distribution of the reflected speckle pattern $P(R_{ij})$ can be obtained by integration over all $m$’s ($m \neq j$):

$$P(R_{ij}) \propto \int \ldots \int P(R_{im}) \prod_{m \neq j} dr_{im}. \quad (A3)$$

For $\beta = 2$ there is no difference between diagonal and off-diagonal elements and the (normalized) distribution for any $i, j$ is given by

$$P(R_{ij}, \beta = 2) = (N - 1)(1 - R_{ij})^{N-2}. \quad (A4)$$

For $\beta = 1$ we obtain:

$$P(R_{ii}, \beta = 1) = \left(\frac{N-1}{2}\right)(1 - R_{ij})^{\frac{N-3}{2}} \quad (A5)$$

and

$$P(R_{ij}, \beta = 1)_{i \neq j} = \left(\frac{N-1}{2}\right)^{\frac{N-2}{2}} \text{$_2$F$_1$}\left(\frac{N-1}{2}, 1; N-1; 1 - R_{ij}\right). \quad (A6)$$

where $\text{$_2$F$_1$}$ is the hypergeometric function.

From these distributions, it is easy to obtain a closed expression for the mean values

$$\langle R_{ij}, \beta \rangle = \int R_{ij} P(R_{ij}, \beta) dR_{ij} = \frac{1 + \delta_{ij}\delta_{\beta_1}}{N + \delta_{\beta_1}}. \quad (A7)$$

References


[37] Al'tshuler, B. L., 1985, Pis'ma Zh. Eksp. Teor. Fiz. [JETP Lett.], 41[41], 530[648].


